Proof of Theorem 2 in "Money and the Decentralization of Exchange" *and Some Comments

Theorem 2: There is no trading rule that (i) uses information D.3; (ii) satisfies A; and (iii) completes trading within a single round for any (p, Z, B) satisfying U.

Proof: As the original paper attempted to, I find two economies such that in a certain meeting, (i) the two traders cannot decide which economy they are in; and (ii) they have to make different trades depending on which economy they are in.

There are five traders, $1, 2, ..., 5$, there are five periods, $t = 1, 2, ..., 5$, and there are four types of goods, $1, 2, 3$ and 4. The sequence of meetings is as follows:

$t = 1 : \overline{25}, \overline{34}, \overline{1}.$
$t = 2 : \overline{15}, \overline{23}, \overline{4}.$
$t = 3 : \overline{12}, \overline{45}, \overline{3}.$
$t = 4 : \overline{14}, \overline{35}, \overline{2}.$
$t = 5 : \overline{13}, \overline{24}, \overline{5}.$

Two traders under the same line are meeting with each other. For example, in $t = 1$, trader 2 meets with 5 and trader 3 meets with 4, but trader 1 does not meet with any other trader.

First, each type of good has the same price, so $p = (1, 1, 1, 1)$.

Let N be the matrix of excess demands, where

$$N = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -4 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Each row represents a trader and each column represents a type of good. For example, $N_{12} = -2$ means that trader 1's excess demand for good 2 is -2.

As in the original paper, the endowment of trader i in good j is $[-N_{ij}]^+$. This endowment is the minimum amount necessary to make sure that no trader desires to hold a negative amount of any good. This setup makes the goods as scarce as possible so that completing trades becomes as difficult as possible. (This is a mathematical statement that can be easily proved.)

Below, I go through individual periods. First, trader 5 has zero endowmentl, so he can never trade. In $t = 1$, trader 3 meets with 4, but they both have only good 4. Therefore, they do not make any trade, and no trade occurs in $t = 1$.;

In $t = 2$, traders 2 and 3 can trade. After trading with 2, trader 3 does not trade with anyone until he trades with trader 1 in $t = 5$, and both trader 1 and 3 want to hold zero amount of good 4. Therefore, trader 3 must give 1 unit of good 4 to trader 2 in $t = 2$. In return, trader 2 gives 1 unit of good 1 to trader 3, since it is the only type of good that he has. After $t = 2$, the matrix of excess demands is:

$$\begin{bmatrix} 4 & -2 & -2 & 0 \\ -3 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In $t = 3$, traders 1 and 2 can trade. Trader 2 needs to give 3 units of good 1 to trader 1 because only trader 1 demands good 1; after $t = 3$, trader 2 meets with trader 4 in $t = 5$, so $t = 3$ is the last chance for trader 2 to pass his stock of good 1 to trader 1.

In return, trader 1 needs to give 3 units of goods 2 and 3. In $t = 5$, he needs to give 1 unit of good 2 to trader 3 because trader 3 cannot receive that good from anyone else. Also, trader 4 will have only good 4 in $t = 4$, so trader 1 must hold 1 unit of good 2 at the end of $t = 3$. Therefore, trader 1 must give 1 unit of good 2 and 2 units of good 3 to trader 2.

To complete the proof, I construct a similar but different economy. The price vector is the same, $p = (1, 1, 1, 1)$. The matrix of excess demands, denoted by M, is

$$M = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -4 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

M can be obtained from N by exchanging row 3 with row 4. The endowment is also given in the same way: The endowment of trader i in good j is $[-M_{ij}]^+$.

Going through the same steps as I did for the other economy, I find that trader 1 needs to give 2 units of good 2 and 1 unit of good 3 to trader 2 in $t = 3$. In the previous economy, trader 1 needed to give 1 unit of good 2 and 2 units of good 3 to trader 2.

Finally, I show that traders 1 and 2 cannot know whether they are in economy N or M in $t = 3$. In both economies, the history of trader 1's excess demands is simply a repeat of $(4, -2, -2, 0)$ because he never trades before $t = 3$. In both economies, the history of trader 2's excess demands is: Before $t = 2$, the excess demand is $(-4, 1, 1, 2)$, and after $t = 2$, it is $(-3, 1, 1, 1)$.

This completes the proof.

In the proof, traders 1 and 2 have difficulty deciding what to do in $t = 3$. Trader 1 needs to pay 3 units of goods to trader 2. It makes sense to pay at least 1 unit of each good to trader 2, but it is not clear in which good the remaining 1 unit should be paid. If trader 1 pays in the wrong type of good, that good will end up with a trader that does not want it. With decentralization, trader 1 has no way to determine which type of good is the right payment.

Such a difficulty can be avoided if there exists a good that can always be traded to make extra payments between traders. Money may serve as an example of such a good in the real world and theorem 4 in the original paper formalizes this idea.